# An Optimal Policy for a Two Depot Inventory Problem with Stock Transfer

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**Abstract:** Multiple depot inventory systems with stock transfer are used by many companies especially when demand is high relative to storage capacity. The key issues in such systems are how many of each item to hold at each depot and what to do if there is a demand for an item at a depot that has none of that item in stock. This study was motivated by the inventory problem faced by a UK car part retailer that groups its depots into pairs. The company's policy for dealing with a demand at a depot that cannot be satisfied from local stock is to either transfer the item from the other depot in the group or to place an emergency order. The object of this paper is to characterise an optimal policy for this problem and to propose a method of calculating the parameters of such a policy.

**Keywords:** Inventory systems; Markov decision models; Dynamic programming

## 1 Introduction

Consider a retail company that stocks and sells a single item at two depots. Stocks can be replenished periodically from an outside source, while customer demands arise continuously (according to independent Poisson processes) at the two depots. A customer demand can be satisfied from (i) local stock, (ii) a transfer from the other depot or (iii) an emergency order. The first option involves no additional cost, so it is chosen whenever possible. Although the (unit) cost of an emergency order is larger than that of a transfer, an emergency order may be called for when the inventory at the alternative location is low relative to the remaining time until the next (regular) replenishment opportunity.

The costs in the system consist of regular ordering, emergency ordering and transfer costs — all of which are proportional to the number of units ordered or transferred — and holding costs proportional to the inventory levels at the end of a period. An important assumption is that all orders and transfers occur instantaneously.

The problem is to minimise the expected discounted cost over an infinite horizon. This study aims to find both an optimal policy for reordering at review epochs and an optimal policy for satisfying unmet demand (i.e. demand that cannot be met locally) which for ease of reference we will call a transfer policy. We show that the optimal reorder policy is to order up to given levels in each depot. We also show that the optimal transfer policy has two "control limit" characteristics. Firstly if for a given level of stock in one depot it is optimal to transfer an item of stock to the other depot to deal with unmet demand, then it remains optimal to transfer an item when there is even more stock available in the first depot at that time. Secondly if at a given time it is optimal to transfer an item from one depot to the other in response to unmet demand, then it remains optimal given the same level of stock in the first depot to transfer in response to unmet demand at all times nearer the next replenishment opportunity.

Originally inventory models dealt with one depot only, see for example the early paper of Veinott [14]. Later models involve multi-echelon depots and, more recently, transshipments between depots at the same level. Tagaras and Cohen [12] and Tagaras [11] look at a two location periodic review inventory problem with transshipment and Federgruen and Klein [4] analyse a multi-period model of a multi-location inventory system with transshipment. In their models transshipments are allowed at the end of a period, after all that period's demand has occurred. Robinson [8] uses a stochastic dynamic programming model to characterise the nature of an optimal transshipment policy. Again the total demand is known at some time and transshipments occur after this time. Das [2] performs a one period analysis of supply and redistribution rules for two location inventory systems in which a single transshipment is allowed within a period. Karmarkar [5] and Showers [10] consider multi-period, multi-location inventory models in which transshipment occurs at the beginning of a period only, in anticipation of demand. Axsater [1], Lee [6] and Sherbrooke [9] examine continuous review inventory systems with repairable items and one-for-one stock replenishments in which transshipments can occur in response to stockouts. Ernst and Cohen [3] explore the cost of a one depot inventory system with emergency orders and two types of customer. In contrast we analyse a multi-period, periodic review model of a two location inventory system in which transshipments can occur (in response to stockouts) at any time during a period; the number of transshipments during a period is unlimited; a stockout can be satisfied by either transshipment or emergency order and the total demand in a period is not known when any transshipment occurs. Our approach is to model the problem as a Markov decision process. White [15] outlines some other types of inventory applications which use this approach.

Section 2 formulates the two depot single item inventory problem as a Markov decision process. Section 3 proves structural results about the optimal policy for such a problem and describes how the parameters of this optimal policy vary with the characteristics of the inventory system. Section 4 incorporates these results in a two depot multiple item inventory problem. The connection between the problems for different items is the limit on storage space. When each depot has a common holding cost for every item, one could ensure this limit is satisfied by adjusting the common holding costs. Since in this application the company wished to make efficient use of its full storage, we aim to find common holding costs for which the limit is just satisfied. This is done using a bisection method. Section 5 applies this algorithm to a particular example.

## 2 A model for a two depot single item inventory system

We first specify the notation. Let $\lambda_k$ and $h_k$ denote the demand and holding cost rates at depot $k$, $k = 1, 2$. The ordering cost rates are assumed to be identical for both depots: that of regular orders is $c$, and of emergency orders is $E$ with $E > c$ to avoid the trivial case where no regular orders are needed. The cost of transferring one item from depot 1(2) to depot 2(1) is $T_{1,2}(T_{2,1})$.

We assume each depot has a limited storage capacity, given by $M_k$ ($k = 1, 2$). Finally, future costs are discounted by a factor $\beta$. We scale the units of time so that the time between successive review epochs is 1. A period is a time interval of length 1 starting just after a review epoch and finishing at the next review epoch. Hence the last decision to be made in a period is how many items to order for the next period.

To simplify the Markov decision model we assume that all items in stock at a review epoch can be returned to the supplier and a full refund can be obtained for each item returned. We will show later that this assumption does not affect the optimality of actions relating to the persistent states in the model.

We model the transfer problem as a finite horizon continuous time Markov decision process. The state of the system is described by two factors: $i_1$ the stock level in depot 1 and $i_2$ the stock level in depot 2. The notation $(i_1, i_2)$ will be used to denote the state. Action choices only occur at two instances.

1. In states $(i_1, 0)$, $0 < i_1 \leq M_1$, when there is a demand at depot 2 and the decision is whether to transfer an item from depot 1 or to place an emergency order.
2. In states $(0, i_2)$, $0 < i_2 \leq M_2$, when there is a demand at depot 1 and the decision is whether to transfer an item from depot 2 or to place an emergency order.

Note that a demand at either depot when the state is $(0, 0)$ does not involve any decision as this demand plus all subsequent demands must be satisfied by emergency orders. If there is a demand at a depot when the system is in state $(0, 0)$ and the time until the next review epoch is $t > 0$, the expected total cost until the next review epoch is

$$E + \sum_{n=0}^{\infty} e^{-(\lambda_1+\lambda_2)t} \frac{((\lambda_1 + \lambda_2)t)^n}{n!} nE = E + (\lambda_1 + \lambda_2)tE$$

Let $W_1(i_1, i_2)$ be the minimum expected total cost until the next review epoch given that the system is in state $(i_1, i_2)$ and the time until the next review epoch is 1 (i.e. a review epoch has just passed). Let $w_t(i_1, i_2)$ be the minimum expected total cost until the next review epoch given that the system is in state $(i_1, i_2)$, there is an unmet demand at one of the depots (so either $i_1 = 0$ or $i_2 = 0$ or both) and the time until the next review epoch is $t$. The results of Miller [7] show that such a function exists. Let $w_t^T(i_1, i_2)$ and $w_t^E(i_1, i_2)$ be the minimum expected total costs until the next review epoch given that the time until the next review epoch is $t$, the system is in state $(i_1, i_2)$ and there is an unmet demand at one of the depots which is satisfied by a transfer (so either $i_1 = 0$ or $i_2 = 0$ but not both) and an emergency order (so either $i_1 = 0$ or $i_2 = 0$ or both) respectively. Let $W_0(i_1, i_2)$ be the total cost of being in state $(i_1, i_2)$ immediately before a review epoch. We want to choose appropriate values for $W_0(.,.)$ so that $W_1(i_1, i_2)$ can be interpreted as the minimum expected total cost of satisfying the demand in a period when the stock levels in depot 1 and depot 2 at the beginning of the period are $i_1$ and $i_2$ respectively. Since a full refund can be obtained for surplus stock at a review epoch and the order cost is linear with no fixed cost, the optimal decision at a review epoch is independent of the stock level just before a review epoch. Hence we need only consider one state of the system at a review epoch. It is convenient to use the state $(0,0)$. The appropriate values for $W_0(.,.)$ are then the holding costs less the return from selling off the remaining inventory at depot 1 and depot 2. Since this problem is a finite horizon continuous time Markov decision process with finite state space and finite action spaces, the optimal value function satisfies the following optimality equation.

$$\begin{aligned}
W_1(i_1,i_2) &= \int_0^1 \lambda_1 e^{-\lambda_1 t}\frac{(\lambda_1 t)^{i_1}}{i_1!}\sum_{k=0}^{i_2} e^{-\lambda_2 t}\frac{(\lambda_2 t)^k}{k!} w_{1-t}(0,i_2-k)\mathrm{d}t \\
&+ \int_0^1 \lambda_2 e^{-\lambda_2 t}\frac{(\lambda_2 t)^{i_2}}{i_2!}\sum_{k=0}^{i_1} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!} w_{1-t}(i_1-k,0)\mathrm{d}t \\
&+ \sum_{k_1=0}^{i_1}\sum_{k_2=0}^{i_2} e^{-\lambda_1}\frac{(\lambda_1)^{k_1}}{k_1!} e^{-\lambda_2}\frac{(\lambda_2)^{k_2}}{k_2!} W_0(i_1-k_1,i_2-k_2) \\
w_t(0,0) &= E + (\lambda_1+\lambda_2)tE + W_0(0,0) \\
w_t(i_1,0) &= \min\left\{w_t^T(i_1,0), w_t^E(i_1,0)\right\} \text{ for } 0 < i_1 \le M_1 \\
w_t(0,i_2) &= \min\left\{w_t^T(0,i_2), w_t^E(0,i_2)\right\} \text{ for } 0 < i_2 \le M_2 \\
W_0(i_1,i_2) &= h_1 i_1 + h_2 i_2 - c i_1 - c i_2 \qquad (1)
\end{aligned}$$

The expression for $W_1(i_1,i_2)$ is obtained by conditioning on the time until the first unmet demand. The first and second integrals deal with the cases of the first unmet demand occurring at depot 1 and depot 2 respectively. The third term deals with the case of no unmet demand occurring before the next review epoch. Since demand is modelled as a Poisson process, the time until the $i^{\text{th}}$ demand occurs at depot $k$ has an Erlang-$i$ distribution with parameter $\lambda_k$ ($k = 1$ or 2).

Conditioning on the time until the next demand at depot 2 gives

$$\begin{aligned}
w_t^E(i_1,0) &= E + \int_0^t \lambda_2 e^{-\lambda_2 s}\left\{\sum_{k=0}^{i_1} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!} w_{t-s}(i_1-k,0)\right. \\
&+ \left.\sum_{k=i_1+1}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\Big((k-i_1)E + w_{t-s}(0,0)\Big)\right\}\mathrm{d}s \\
&+ e^{-\lambda_2 t}\left\{\sum_{k=0}^{i_1} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!} W_0(i_1-k,0) + \sum_{k=i_1+1}^{\infty} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}\Big((k-i_1)E + W_0(0,0)\Big)\right\} \qquad (2)
\end{aligned}$$

It is easy to verify that

$$w_t^T(i_1,0) = T_{1,2} - E + w_t^E(i_1-1,0) \text{ for } 0 < i_1 \le M_1$$

Similar expressions exist for $w_t^E(0,i_2)$ and $w_t^T(0,i_2)$ with the roles of depot 1 and depot 2 reversed.

We model the problem of reordering at review epochs as an infinite horizon discounted Markov decision process. As noted above there is only one state in this problem because of the choice of the terminal values $W_0(i_1,i_2)$ in (1). This state is $(0,0)$ — no items in depot 1 and no items in depot 2. The decision is how many items to have in stock at each depot at the beginning of the next period. If the decision is to have $i_1$ items at depot 1 and $i_2$ items at depot 2 at the beginning of the next period, the minimum expected total cost of satisfying the demand during the next period is $W_1(i_1, i_2)$. Let $V(0,0)$ be the minimum expected discounted cost over an infinite horizon. Since the problem is an infinite horizon discounted Markov decision process with finite state space and finite action spaces, the optimal value function satisfies the following optimality equation.

$$V(0,0) = \min_{\substack{i_1 : 0 \le i_1 \le M_1 \\ i_2 : 0 \le i_2 \le M_2}} \left\{ ci_1 + ci_2 + \beta\left(W_1(i_1, i_2) + V(0,0)\right)\right\} \tag{3}$$

# 3 Characterisation of an optimal policy

**Proposition 1** There exist non-negative integers $S_1$ and $S_2$ such that the optimal reorder policy is to order up to stock level $S_1$ at depot 1 and to order up to stock level $S_2$ at depot 2.

**Proof** Since the problem has been formulated as a single state Markov decision process, a policy is defined by a single decision. $S_1$ and $S_2$ are the values of $i_1$ and $i_2$ that minimise the right hand side of (3).

$\diamondsuit$

To prove structural results about the optimal transfer policy, we introduce value iteration. For the Markov decision process formulation of the transfer problem the standard value iteration algorithm is as follows (where the second subscript denotes the iteration number). The choice of starting values for the iteration is arbitrary. We have chosen them to ensure that $w_{t,0}(.,0)$ and $w_{t,0}(0,.)$ are of the form required in later proofs.

$$W_{1,0}(i_1, i_2) = 0, \quad w_{t,0}(i_1, 0) = -ci_1, \quad w_{t,0}(0, i_2) = -ci_2$$

$$W_{0,n}(i_1, i_2) = h_1 i_1 + h_2 i_2 - ci_1 - ci_2 \text{ for } n \ge 0$$

$$\begin{aligned}
W_{1,n+1}(i_1, i_2) = & \int_0^1 \lambda_1 e^{-\lambda_1 t} \frac{(\lambda_1 t)^{i_1}}{i_1!} \sum_{k=0}^{i_2} e^{-\lambda_2 t} \frac{(\lambda_2 t)^k}{k!} w_{1-t,n}(0, i_2 - k) \mathrm{d}t \\
& + \int_0^1 \lambda_2 e^{-\lambda_2 t} \frac{(\lambda_2 t)^{i_2}}{i_2!} \sum_{k=0}^{i_1} e^{-\lambda_1 t} \frac{(\lambda_1 t)^k}{k!} w_{1-t,n}(i_1 - k, 0) \mathrm{d}t \\
& + \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} e^{-\lambda_1} \frac{(\lambda_1)^{k_1}}{k_1!} e^{-\lambda_2} \frac{(\lambda_2)^{k_2}}{k_2!} W_{0,n}(i_1 - k_1, i_2 - k_2)
\end{aligned}$$

$$w_{t,n+1}(0,0) = E + (\lambda_1 + \lambda_2) t E + W_{0,n}(0,0)$$

$$w_{t,n+1}(i_1, 0) = \min\left\{ w^T_{t,n+1}(i_1, 0), w^E_{t,n+1}(i_1, 0)\right\} \text{ for } 0 < i_1 \le M_1$$

$$\begin{aligned}
w^E_{t,n+1}(i_1, 0) = E + & \int_0^t \lambda_2 e^{-\lambda_2 s} \left\{ \sum_{k=0}^{i_1} e^{-\lambda_1 s} \frac{(\lambda_1 s)^k}{k!} w_{t-s,n}(i_1 - k, 0) \right. \\
& \left. + \sum_{k=i_1+1}^{\infty} e^{-\lambda_1 s} \frac{(\lambda_1 s)^k}{k!} \left((k - i_1)E + w_{t-s,n}(0,0)\right) \right\} \mathrm{d}s
\end{aligned}$$

$$+ e^{-\lambda_2 t}\left\{\sum_{k=0}^{i_1} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}W_{0,n}(i_1-k,0) + \sum_{k=i_1+1}^{\infty} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}\Big((k-i_1)E + W_{0,n}(0,0)\Big)\right\} \quad (4)$$

$$w^T_{t,n+1}(i_1,0) = T_{1,2} - E + w^E_{t,n+1}(i_1-1,0) \text{ for } 0 < i_1 \le M_1 \quad (5)$$

with similar expressions for $w_{t,n+1}(0,i_2)$, $w^E_{t,n+1}(0,i_2)$ and $w^T_{t,n+1}(0,i_2)$

**Lemma 1** For the continuous time finite horizon Markov decision process formulation of the transfer problem, the functions $w_{t,n}(i_1,0)$, $w_{t,n}(0,i_2)$ and $W_{1,n}(i_1,i_2)$ (where $0 \le i_1 \le M_1$, $0 \le i_2 \le M_2$ and $0 < t < 1$) from the standard value iteration algorithm converge to the optimal value functions $w_t(i_1,0)$, $w_t(0,i_2)$ and $W_1(i_1,i_2)$.

**Proof** Define

$$\Big\| w_\cdot(.,0) - w_{\cdot,n}(.,0)\Big\| = \max_{i\,:\,0\le i\le M_1}\left\{\sup_{t\,:\,0<t<1}\Big|w_t(i,0) - w_{t,n}(i,0)\Big|\right\}$$

Since $W_{0,n}(i_1,i_2) = W_0(i_1,i_2)$ for all $n$, it can be shown (see appendix) that

$$\Big\| w_\cdot(.,0) - w_{\cdot,n+1}(.,0)\Big\| \le \Big(1 - e^{-\lambda_2}\Big)\Big\| w_\cdot(.,0) - w_{\cdot,n}(.,0)\Big\| < \Big\| w_\cdot(.,0) - w_{\cdot,n}(.,0)\Big\|$$

Hence $w_{t,n}(i_1,0)$ converges to $w_t(i_1,0)$ as $n$ tends to infinity. A similar argument can be used to show that $w_{t,n}(0,i_2)$ converges to $w_t(0,i_2)$ as $n$ tends to infinity. Finally since $W_{1,n+1}(i_1,i_2)$ is expressed in terms of $w_{t,n}(.,0)$, $w_{t,n}(0,.)$ and $W_{0,n}(.,.)$, $W_{1,n}(i_1,i_2)$ must also converge to a limit as $n$ tends to infinity. On comparing the forms of the standard value iteration algorithm and the optimality equation for the transfer problem, it is apparent that this limit must be $W_1(i_1,i_2)$.

$\diamondsuit$

The remaining results of this section describe the form of an optimal policy for deciding whether to transfer or place an emergency order when there is a demand at a depot that cannot be met from local stock.

**Lemma 2** For all $n \ge 0$, $w^E_{t,n}(i,0)$ and $w_{t,n}(i,0)$ are convex in $i$ and submodular in $(t,i)$ and $w^E_{t,n}(0,j)$ and $w_{t,n}(0,j)$ are convex in $j$ and submodular in $(t,j)$. (Note that $w_{t,n}(i,0)$ is submodular in $(t,i)$ if and only if $w_{r,n}(\ell+1,0) - w_{r,n}(\ell,0) \le w_{s,n}(\ell+1,0) - w_{s,n}(\ell,0)$ when $s \le r$, see Topkis [13].)

**Proof** The proof is by induction on $n$, the iteration number. See the appendix for details.

$\diamondsuit$

**Lemma 3** There exist real values $\tau^1_{1,n} \le \tau^1_{2,n} \le \ldots \le \tau^1_{M_1,n}$ such that, at iteration $n$ of the value iteration algorithm, the minimising action in state $(i,0)$ when there is an unmet demand at depot 2 and $t$ to go until the next review epoch, is to transfer an item from depot 1 to depot 2 if $t \le \tau^1_{i,n}$ and to place an emergency order otherwise.

Similarly there exist real values $\tau^2_{1,n} \le \tau^2_{2,n} \le \ldots \le \tau^2_{M_2,n}$ such that, at iteration $n$ of the value iteration algorithm, the minimising action in state $(0,j)$ when there is an unmet demand at depot

1 and $t$ to go until the next review epoch, is to transfer an item from depot 2 to depot 1 if $t \leq \tau^2_{j,n}$ and to place an emergency order otherwise.

**Proof** Note that $w_{t,n}(0,0) = w^E_{t,n}(0,0)$. For $i > 0$, $w_{t,n}(i,0) = w^T_{t,n}(i,0)$ if and only if $w^E_{t,n}(i,0) \geq w^T_{t,n}(i,0)$ and, using (5), $w^E_{t,n}(i,0) \geq w^T_{t,n}(i,0)$ if and only if $w^E_{t,n}(i,0) - w^E_{t,n}(i-1,0) \geq T_{1,2} - E$. By Lemma 2 $w^E_{t,n}(i,0)$ is submodular in $(t,i)$, so $w^E_{t,n}(i,0) - w^E_{t,n}(i-1,0)$ is non-increasing in $t$. Hence there exists $\tau^1_{i,n}$ such that $w_{t,n}(i,0) = w^T_{t,n}(i,0)$ if $0 < t \leq \tau^1_{i,n}$ and $w_{t,n}(i,0) = w^E_{t,n}(i,0)$ if $t > \tau^1_{i,n}$. By Lemma 2 $w^E_{t,n}(i,0)$ is convex in $i$, so $w^E_{t,n}(i,0) - w^E_{t,n}(i-1,0)$ is non-decreasing in $i$. Hence $\tau^1_{i+1,n} \geq \tau^1_{i,n}$.

A similar argument can be used to prove the existence of $\tau^2_{1,n}, \tau^2_{2,n}, \dots, \tau^2_{M_2,n}$.

$\diamondsuit$

**Proposition 2** There exist real values $\tau^1_1 \leq \tau^1_2 \leq \dots \leq \tau^1_{M_1}$ such that the minimising action in state $(i,0)$ when there is an unmet demand at depot 2 and $t$ to go until the next review epoch is to transfer an item from depot 1 to depot 2 if $t \leq \tau^1_i$ and to place an emergency order otherwise.

Similarly there exist real values $\tau^2_1 \leq \tau^2_2 \leq \dots \leq \tau^2_{M_2}$ such that the minimising action in state $(0,j)$ when there is an unmet demand at depot 1 and $t$ to go until the next review epoch is to transfer an item from depot 2 to depot 1 if $t \leq \tau^2_j$ and to place an emergency order otherwise.

**Proof** By Lemma 1 the standard value iteration algorithm for the transfer problem converges to the optimal value function, so the optimal value function has the properties described in Lemma 2. This result then follows by repeating the argument of Lemma 3.

$\diamondsuit$

Figure 1 shows a policy for transferring items from depot 2 to depot 1 that has the property described in Proposition 2.

Propositions 1 and 2 combine to give the optimal policy for the overall problem. Under this policy, when the state of the system at a review epoch is $(i_1, i_2)$ we sell off $i_1$ items from depot 1 and $i_2$ items from depot 2 and then immediately order $S_1$ items for depot 1 and $S_2$ items for depot 2. If $i_1 \leq S_1$ and $i_2 \leq S_2$, this is equivalent to ordering $S_1 - i_1$ items for depot 1 and $S_2 - i_2$ items for depot 2 and this action is feasible for the case in which items cannot be returned to the supplier. This leads to the following proposition.

**Proposition 3** When surplus stock cannot be returned to the supplier and there are $i_1 \leq S_1$ items at depot 1 and $i_2 \leq S_2$ items at depot 2, where $S_1$ and $S_2$ are defined in Proposition 1, the optimal reorder policy is to order $S_1 - i_1$ items for depot 1 and $S_2 - i_2$ items for depot 2 and the optimal transfer policy is of the form defined in Proposition 2.

**Proof** If surplus stock can be returned to the supplier and the system is allowed to be in any state at a review epoch then, when the policy defined by Propositions 1 and 2 is applied, the set of persistent states in the Markov chain is $\{(i_1, i_2) : 0 \leq i_1 \leq S_1, 0 \leq i_2 \leq S_2\}$. If we restrict the problem so that surplus stock cannot be returned to the supplier, no new actions are introduced and the optimal actions for the persistent states in the unrestricted problem remain feasible. Hence for these states the optimal actions in the restricted and unrestricted problems are the same. (Note that the policy is no longer feasible for the transient states in the restricted problem.)

$\diamondsuit$

We use the following method to determine an optimal policy for a particular instance of the problem. We discretise time and calculate, working backwards in time directly from the optimality equation for the transfer problem, $w_t(i_1, 0)$, $w_t(0, i_2)$ and $W_1(i_1, i_2)$ for $0 \leq i_1 \leq M_1$, $0 \leq i_2 \leq M_2$ and $t$ a point in the discretisation. $\tau_i^1$ and $\tau_j^2$ can be approximated by the latest points in the discretisation for which a transfer optimises $w_t(i, 0)$ and $w_t(0, j)$ respectively. $S_1$ and $S_2$ can then be determined from the optimality equation for the reordering problem by complete enumeration. The computational effort required to determine an optimal policy increases linearly with the capacity of depot 1, the capacity of depot 2 and the number of points in the discretisation of time.

If the model is modified so that holding costs are incurred continuously or future costs are discounted continuously, the optimal policy may not be of the form described in Proposition 3. It will still be the case that there exist $S_1$ and $S_2$ such that when there are $i_1 \leq S_1$ items at depot 1 and $i_2 \leq S_2$ items at depot 2, the optimal reorder policy is to order up to $S_1$ at depot 1 and up to $S_2$ at depot 2. Also if it is optimal to transfer when there is a demand at depot 2 and the system is in state $(i, 0)$ then it is optimal to transfer when there is a demand at depot 2 and the system is in state $(\ell, 0)$ for $i < \ell \leq S_1$ (a similar result holds for depot 1). However the threshold times $\tau_i^k$ described in Proposition 2 may not exist. This is because in both cases there are reasons for favouring transfers over emergency orders near the beginning of a period. If holding costs are incurred continuously then the earlier in the period transfers are carried out, the lower the period

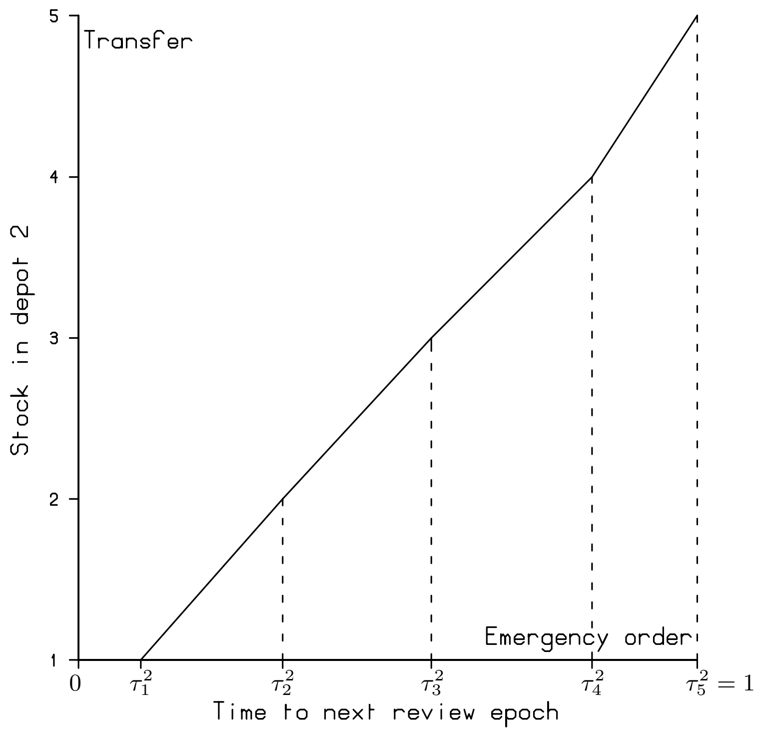


Figure 1: An illustration of a possible transfer policy from depot 2 to depot 1

holding costs will be. With continuous discounting a transfer near the beginning of a period may be attractive because an early transfer followed by a much later (and hence heavily discounted) emergency order may be a lot cheaper than an early emergency order followed by a much later transfer.

If the model is modified so that replenishment lead times or fixed order costs are non-zero, it is not possible to define terminal values $W_0(.,.)$ for which $W_1(i_1, i_2)$ can be interpreted as the minimum expected total cost of satisfying the demand in a period when the stock levels in depot 1 and depot 2 at the beginning of the period are $i_1$ and $i_2$ respectively. This means that the problems of reordering at review epochs and of satisfying unmet demand within a period cannot be separated and hence the analysis presented in this paper is not appropriate. However it is worth noting that if the fixed order cost is small relative to the other costs then the transfer policy described in this paper will be a good approximation to the optimal transfer policy.

The optimal policy is related to the problem characteristics in the following ways. If the demand at either depot is increased then the values of $S_1$ and $S_2$ will increase (subject to the constraints imposed by the depot capacities) to balance the increased likelihood of a stockout. If the emergency order cost or transfer costs are increased then the values of $S_1$ and $S_2$ will increase to reduce the likelihood of these costs being incurred. If the depot capacity constraints are not limiting, the optimal order up to level for a depot does not exceed the level that would be expected if that depot was considered in isolation, because the stock in the other depot can be used as safety stock. Transfers from depot 1 to depot 2 will first occur later in a period if:

- the demand at depot 1 is increased, because of the increased likelihood of a wasted transfer (i.e. the transfer of an item that could subsequently have been used at depot 1);
- the demand at depot 2 is increased, because of the increased likelihood of a further opportunity to transfer the item arising;
- the transfer cost from depot 1 to depot 2 is increased, because of the higher expected cost of a wasted transfer;
- the emergency order cost is reduced, because of the relatively higher expected cost of a wasted transfer.

# 4 A model for a two depot multiple item inventory system

We now extend the model described in section 2 to the case of a multiple item inventory system. $M_k$ is now the total storage capacity in depot $k$. For ease of exposition we will assume that for depot $k$ the upper limit on the number of items of a particular type that can be stored is $M_k$ and the holding cost is $h_k$, independent of the item type. These restrictions could be relaxed in the following ways. Firstly a volume could be assigned to each item type and the holding cost could then be interpreted as a cost per unit of volume occupied. Secondly if there is an alternative upper limit on the number of items of a particular type that can be stored in depot $k$, this limit could replace $M_k$ in the optimality equations for that item type. All the other parameters in the two depot single item model can be specific to the item type (i.e. demand rates, regular ordering cost, transfer costs and emergency ordering cost).

Suppose there are $m$ item types in the model. In the motivating problem of a car part retailer the storage capacities $M_k$ are so limited and the number of item types $m$ is so large relative to the real holding costs $h_k$, that the optimal policy is to fully use the total storage capacity. We will require this total use of capacity in our optimal policy.

For given depot holding costs, $h_1$ and $h_2$ at depot 1 and depot 2 respectively, we can solve the two depot single item model for each item type independently resulting in a series of $m$ optimal reorder policies $(S_1^1, S_2^1)$, $(S_1^2, S_2^2)$, $\ldots$, $(S_1^m, S_2^m)$ (where $(S_1^j, S_2^j)$ is an optimal reorder policy for item type $j$). Let $\sigma_k(h_1, h_2) = \sum_{j=1}^m S_k^j$. Our solution method is based on the observation that when the holding cost at a depot is increased, the optimal reorder levels at that depot will not increase. A starting point for the method is obtained by finding depot holding costs $(\underline{h}_1, \underline{h}_2)$ for which individual optimal reorder policies more than fill both depots and depot holding costs $(\overline{h}_1, \overline{h}_2)$ for which individual optimal reorder policies fill neither depot (i.e. $\sigma_k(\underline{h}_1, \underline{h}_2) > M_k$ and $\sigma_k(\overline{h}_1, \overline{h}_2) < M_k$ for $k = 1$ and 2). If no such holding costs can be found, the method fails and there are no optimal reorder policies that would fill both depots. It seems likely that there will exist depot holding costs $(h_1, h_2)$ satisfying $\underline{h}_k \leq h_k \leq \overline{h}_k$ for $k = 1$ and 2 for which individual optimal reorder policies exactly fill both depots and our method aims to find such a point using a bisection method.

The solution method bisects the ranges of the depot 1 and depot 2 holding costs alternately until individual optimal reorder policies exactly fill both depots or the number of bisections performed reaches the maximum number permitted. The procedure for bisecting the range of the depot 1 holding cost is as follows.

1. Let $\tilde{h}_1 = (\underline{h}_1 + \overline{h}_1)/2$
2. If $\sigma_1(\tilde{h}_1, \underline{h}_2) \geq M_1$ and $\sigma_2(\tilde{h}_1, \underline{h}_2) \geq M_2$ then let $\underline{h}_1 = \tilde{h}_1$ and check stopping criteria
3. If $\sigma_1(\tilde{h}_1, \overline{h}_2) \leq M_1$ and $\sigma_2(\tilde{h}_1, \overline{h}_2) \leq M_2$ then let $\overline{h}_1 = \tilde{h}_1$ and check stopping criteria
4. If $\sigma_1(\tilde{h}_1, \underline{h}_2) \geq M_1$ and $\sigma_1(\tilde{h}_1, \overline{h}_2) \leq M_1$ then search for $\tilde{h}_2$ satisfying $\underline{h}_2 < \tilde{h}_2 < \overline{h}_2$ and either (A) $\sigma_1(\tilde{h}_1, \tilde{h}_2) \geq M_1$ and $\sigma_2(\tilde{h}_1, \tilde{h}_2) \geq M_2$ or (B) $\sigma_1(\tilde{h}_1, \tilde{h}_2) \leq M_1$ and $\sigma_2(\tilde{h}_1, \tilde{h}_2) \leq M_2$
   (a) if (A) is satisfied then let $\underline{h}_1 = \tilde{h}_1$, $\underline{h}_2 = \tilde{h}_2$ and check stopping criteria
   (b) if (B) is satisfied then let $\overline{h}_1 = \tilde{h}_1$, $\overline{h}_2 = \tilde{h}_2$ and check stopping criteria
   (c) if search is unsuccessful then continue from 5 below
5. If $\sigma_1(\tilde{h}_1, \underline{h}_2) < M_1$ then search for $h_1'$ satisfying $\underline{h}_1 \leq h_1' < \tilde{h}_1$, $\sigma_1(h_1', \underline{h}_2) \geq M_1$ and $\sigma_2(h_1', \underline{h}_2) \geq M_2$; let $\underline{h}_1 = h_1'$ and check stopping criteria
6. If $\sigma_1(\tilde{h}_1, \overline{h}_2) > M_1$ then search for $h_1'$ satisfying $\tilde{h}_1 < h_1' \leq \overline{h}_1$, $\sigma_1(h_1', \overline{h}_2) \leq M_1$ and $\sigma_2(h_1', \overline{h}_2) \leq M_2$; let $\overline{h}_1 = h_1'$ and check stopping criteria

Stopping criteria: both depots filled exactly or maximum number of bisections reached

The procedure for bisecting the range of the depot 2 holding cost is similar. Note that even if there exist depot holding costs for which both depots are exactly filled when individual optimal reorder policies are used, the method described above is not guaranteed to find them.

# 5 An example of a two depot two item inventory system

Consider the two depot two item inventory system with the following parameters.

*Discount factor*
$\beta = 0.995$

*Depot specific parameters*
$M_1 = M_2 = 10$

*Item specific parameters*

| Parameter | $\lambda_1$ | $\lambda_2$ | $c$ | $T_{1,2}$ | $T_{2,1}$ | $E$ |
|---|---|---|---|---|---|---|
| Item 1 | 4.0 | 2.0 | 1.0 | 0.8 | 0.8 | 2.0 |
| Item 2 | 2.5 | 2.0 | 1.0 | 0.5 | 0.5 | 2.0 |

With depot holding costs $h_1 = h_2 = 0.005$ (0.5% of the capital cost of inventory each period), the optimal reorder policy for item type 1 is to order up to 9 items at depot 1 and 6 items at depot 2 and the optimal reorder policy for item type 2 is to order up to 6 items at depot 1 and 5 items at depot 2. When combined these policies violate the capacity constraint at both depots and so they are not feasible reorder policies for the two item problem. If the capacity constraints were dropped, these reorder policies would be optimal and, when combined with optimal transfer policies, would lead to a minimum expected discounted infinite horizon cost of 2081.96. When the capacity constraints are imposed, the optimal reorder policy is to order up to 6 of item 1 and 4 of item 2 at depot 1 and to order up to 5 of item 1 and 5 of item 2 at depot 2. The minimum expected discounted infinite horizon cost in this case is 2113.57. Hence capacity for 5 additional units at depot 1 and 1 additional unit at depot 2 has value 31.61.

Applying the method described in section 4 to the above problem gives the following results.

*Holding costs*
Individual optimal reorder policies exactly fill both depots when $h_1 = 0.1250$ and $h_2 = 0.0312$

*Optimal reorder policies*
For item 1 order up to 6 at depot 1 and 5 at depot 2.
For item 2 order up to 4 at depot 1 and 5 at depot 2.

*Optimal transfer policies from depot 1 to depot 2*

| Stock in depot 1 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Threshold time for item 1 | 0.07 | 0.24 | 0.42 | 0.61 | 0.80 | 1.00 |
| Threshold time for item 2 | 0.26 | 0.58 | 0.90 | 1.00 | — | — |

*Optimal transfer policies from depot 2 to depot 1*

| Stock in depot 2 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Threshold time for item 1 | 0.10 | 0.33 | 0.57 | 0.83 | 1.00 |
| Threshold time for item 2 | 0.27 | 0.61 | 0.96 | 1.00 | 1.00 |

Note that figure 1 shows the optimal policy for transferring item 1 from depot 2 to depot 1.

The holding costs for which individual optimal reorder policies exactly fill both depots are not unique and a sensitivity analysis for this example shows that these costs can vary by 35%. However the values of the holding costs found in this example give an indication that depot 1 is more constrained than depot 2 and therefore would be the obvious one in which to expand storage capacity. Comparing the optimal transfer policies for item 1 we see that, at all stock levels, transfers from depot 2 to depot 1 occur earlier in the period than transfers from depot 1 to depot 2. This is because the arrival rate of customers requiring item 1 is greater at depot 1 than at depot 2. In general, all other factors being equal, a slow moving part will be transferred earlier in the period than a fast moving part. Transfers early in the period will also be favoured when the emergency cost plus the holding cost is large compared to the transfer cost plus the unit cost.

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# Appendix

## Further details of the proof of Lemma 1

Consider (2) and (4).

$$\begin{aligned}
&\left|w_t^E(i_1,0) - w_{t,n+1}^E(i_1,0)\right| \\
&= \left|\int_0^t \lambda_2 e^{-\lambda_2 s}\left(\sum_{k=0}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\left(w_{t-s}(\max\{i_1-k,0\},0) - w_{t-s,n}(\max\{i_1-k,0\},0)\right)\right)\mathrm{d}s\right. \\
&\qquad \left. + e^{-\lambda_2 t}\left(\sum_{k=0}^{\infty} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}\left(W_0(\max\{i_1-k,0\},0) - W_{0,n}(\max\{i_1-k,0\},0)\right)\right)\right| \\
&= \left|\int_0^t \lambda_2 e^{-\lambda_2 s}\left(\sum_{k=0}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\left(w_{t-s}(\max\{i_1-k,0\},0) - w_{t-s,n}(\max\{i_1-k,0\},0)\right)\right)\mathrm{d}s\right| \\
&\hspace{25em} \text{since } W_{0,n}(i_1,i_2) = W_0(i_1,i_2) \\
&\le \left\|w_\cdot(.,0) - w_{\cdot,n}(.,0)\right\| \int_0^t \lambda_2 e^{-\lambda_2 s}\left(\sum_{k=0}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\right)\mathrm{d}s \\
&= \left(1-e^{-\lambda_2 t}\right)\left\|w_\cdot(.,0) - w_{\cdot,n}(.,0)\right\|
\end{aligned}$$

$$\left|w_t^T(i_1,0) - w_{t,n+1}^T(i_1,0)\right| = \left|w_t^E(i_1-1,0) - w_{t,n+1}^E(i_1-1,0)\right| \le \left(1-e^{-\lambda_2 t}\right)\left\|w_\cdot(.,0) - w_{\cdot,n}(.,0)\right\|$$

$$\begin{aligned}
&\left|w_t(i_1,0) - w_{t,n+1}(i_1,0)\right| = \left|\min\left\{w_t^T(i_1,0), w_t^E(i_1,0)\right\} - \min\left\{w_{t,n+1}^T(i_1,0), w_{t,n+1}^E(i_1,0)\right\}\right| \\
&\le \left|\max\left\{w_t^T(i,0) - w_{t,n+1}^T(i,0), w_t^E(i,0) - w_{t,n+1}^E(i,0)\right\}\right| \\
&\le \left(1-e^{-\lambda_2 t}\right)\left\|w_\cdot(.,0) - w_{\cdot,n}(.,0)\right\|
\end{aligned}$$

Hence $\left\|w_\cdot(.,0) - w_{\cdot,n+1}(.,0)\right\| \le \left(1-e^{-\lambda_2}\right)\left\|w_\cdot(.,0) - w_{\cdot,n}(.,0)\right\|$

## Proof of Lemma 2

We will prove that, for all $n \geq 0$, $w^E_{t,n}(i,0)$ and $w_{t,n}(i,0)$ are convex in $i$ and submodular in $(t,i)$. A similar argument can be used to prove that, for all $n \geq 0$, $w^E_{t,n}(0,j)$ and $w_{t,n}(0,j)$ are convex in $j$ and submodular in $(t,j)$. Define

$$\begin{aligned}
x_{t,n}(i,0) &= w_{t,n}(i+1,0) - w_{t,n}(i,0) \text{ for } 0 \leq i < M_1 \\
x^T_{t,n}(i,0) &= w^T_{t,n}(i+1,0) - w^T_{t,n}(i,0) \text{ for } 0 < i < M_1 \\
x^E_{t,n}(i,0) &= w^E_{t,n}(i+1,0) - w^E_{t,n}(i,0) \text{ for } 0 \leq i < M_1 \\
X_{0,n}(i,0) &= W_{0,n}(i+1,0) - W_{0,n}(i,0) \text{ for } 0 \leq i < M_1
\end{aligned}$$

We will prove the following (note that (ii) and (v) imply Lemma 2 and the other results are required in the proofs of (ii) and (v)).

(i) $X_{0,n}(i,0)$ is non-decreasing in $i$ for $0 \leq i < M_1$ and $X_{0,n}(i,0) \geq -E$

(ii) $x^E_{t,n}(i,0)$ and $x_{t,n}(i,0)$ are non-decreasing in $i$ for $0 \leq i < M_1$ (and hence $w^E_{t,n}(i,0)$ and $w_{t,n}(i,0)$ are convex in $i$)

(iii) $x_{t,n}(0,0) \geq -E$

(iv) $X_{0,n}(i,0) \geq x_{t,n}(i,0)$ for $0 \leq i < M_1$

(v) $x^E_{t,n}(i,0)$ and $x_{t,n}(i,0)$ are non-increasing in $t$ for $0 \leq i < M_1$ (and hence $w^E_{t,n}(i,0)$ and $w_{t,n}(i,0)$ are submodular in $(t,i)$)

**To prove (i)** it is enough to note that $X_{0,n}(i,0) = h_1 - c$. Since $X_{0,n}(i,0)$ is independent of $i$, it is non-decreasing in $i$. Since holding costs are non-negative and $c < E$ by assumption, $X_{0,n}(i,0) \geq -E$.

The proof of (ii), (iii), (iv) and (v) is by induction on $n$, the iteration number. For this proof define $w^E_{t,0}(i,0) = -ci$. Since $x_{t,0}(i,0) = x^E_{t,0}(i,0) = -c$ for $0 < t < 1$ and $0 \leq i < M_1$, the results are true for $n = 0$. Assume the results are true for iteration $n$.

**To prove (ii)** we see from (4) that

$$\begin{aligned}
x^E_{t,n+1}(i,0) &= w^E_{t,n+1}(i+1,0) - w^E_{t,n+1}(i,0) \\
&= \int_0^t \lambda_2 e^{-\lambda_2 s} \left\{ \sum_{k=0}^{i} e^{-\lambda_1 s} \frac{(\lambda_1 s)^k}{k!} x_{t-s,n}(i-k,0) - \sum_{k=i+1}^{\infty} e^{-\lambda_1 s} \frac{(\lambda_1 s)^k}{k!} E \right\} \mathrm{d}s \\
&\quad + e^{-\lambda_2 t} \left\{ \sum_{k=0}^{i} e^{-\lambda_1 t} \frac{(\lambda_1 t)^k}{k!} X_{0,n}(i-k,0) - \sum_{k=i+1}^{\infty} e^{-\lambda_1 t} \frac{(\lambda_1 t)^k}{k!} E \right\}
\end{aligned} \tag{6}$$

By the inductive hypothesis $x_{t,n}(i,0)$ is non-decreasing in $i$ and $x_{t,n}(0,0) \geq -E$, so each term within the integral is non-decreasing in $i$ and hence the integral is non-decreasing in $i$. From (i)

$X_{0,n}(i,0)$ is non-decreasing in $i$ and $X_{0,n}(0,0) \geq -E$, so the second term in the expression is non-decreasing in $i$. Hence $x^E_{t,n+1}(i,0)$ is non-decreasing in $i$.

We see from (5) that $x^T_{t,n+1}(i,0) = x^E_{t,n+1}(i-1,0)$, so $x^T_{t,n+1}(i,0)$ is also non-decreasing in $i$. Moreover $w_{t,n+1}(0,0) = w^E_{t,n+1}(0,0)$ and for $i > 0$ $w_{t,n+1}(i,0) = w^T_{t,n+1}(i,0)$ if and only if $x^E_{t,n+1}(i-1,0) \geq T_{1,2} - E$. Since $x^E_{t,n+1}(i,0)$ is non-decreasing in $i$, there exists $i^*(t,n)$ such that $w_{t,n+1}(i,0) = w^T_{t,n+1}(i,0)$ if $i > i^*(t,n)$ and $w_{t,n+1}(i,0) = w^E_{t,n+1}(i,0)$ if $i \leq i^*(t,n)$. Hence

$$x_{t,n+1}(i,0) = \begin{cases} x^T_{t,n+1}(i,0) \text{ if } i > i^*(t,n) \\ w^T_{t,n+1}(i+1,0) - w^E_{t,n+1}(i,0) = T_{1,2} - E \text{ if } i = i^*(t,n) \\ x^E_{t,n+1}(i,0) \text{ if } i < i^*(t,n) \end{cases}$$

By definition $x^T_{t,n+1}(i,0) = x^E_{t,n+1}(i-1,0) \geq T_{1,2} - E$ if $i > i^*(t,n)$ and $x^E_{t,n+1}(i,0) < T_{1,2} - E$ if $i < i^*(t,n)$, so $x_{t,n+1}(i,0)$ is non-decreasing in $i$ for $0 \leq i < M_1$.

**To prove (iii)** we use the fact that $w_{t,n+1}(0,0) = w^E_{t,n+1}(0,0)$. Hence

$$\begin{aligned} x_{t,n+1}(0,0) &= \min\left\{w^T_{t,n+1}(1,0) - w^E_{t,n+1}(0,0), w^E_{t,n+1}(1,0) - w^E_{t,n+1}(0,0)\right\} \\ &= \min\left\{T_{1,2} - E, x^E_{t,n+1}(0,0)\right\} \end{aligned}$$

From (6)

$$\begin{aligned} x^E_{t,n+1}(0,0) = &\int_0^t \lambda_2 e^{-\lambda_2 s}\left\{e^{-\lambda_1 s} x_{t-s,n}(0,0) - \sum_{k=1}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}E\right\} \mathrm{d}s \\ &+ e^{-\lambda_2 t}\left\{e^{-\lambda_1 t} X_{0,n}(0,0) - \sum_{k=1}^{\infty} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}E\right\} \end{aligned}$$

By the inductive hypothesis $x_{t,n}(0,0) \geq -E$ and from (i) $X_{0,n}(0,0) \geq -E$. It follows that

$$x^E_{t,n+1}(0,0) \geq -E\left(\int_0^t \lambda_2 e^{-\lambda_2 s}\sum_{k=0}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\mathrm{d}s + e^{-\lambda_2 t}\sum_{k=0}^{\infty} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}\right) = -E$$

Hence $x_{t,n+1}(0,0) \geq -E$.

**To prove (iv)** note that $X_{0,n+1}(i,0)$ is independent of $n$ and $i$, so $X_{0,n+1}(i,0) = X_{0,n}(i,0)$. We see from (6) that

$$\begin{aligned} X_{0,n+1}(i,0) - x^E_{t,n+1}(i,0) = &\int_0^t \lambda_2 e^{-\lambda_2 s}\left\{\sum_{k=0}^{i} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\Big(X_{0,n}(i-k,0) - x_{t-s,n}(i-k,0)\Big)\right. \\ &+ \left.\sum_{k=i+1}^{\infty} e^{-\lambda_1 s}\frac{(\lambda_1 s)^k}{k!}\Big(X_{0,n+1}(i,0) + E\Big)\right\}\mathrm{d}s + e^{-\lambda_2 t}\sum_{k=i+1}^{\infty} e^{-\lambda_1 t}\frac{(\lambda_1 t)^k}{k!}\Big(X_{0,n+1}(i,0) + E\Big) \end{aligned}$$

From (i) $X_{0,n+1}(i,0) \geq -E$ and by the inductive hypothesis $X_{0,n}(i-k,0) \geq x_{t-s,n}(i-k,0)$, so the expression is non-negative. Hence $X_{0,n+1}(i,0) \geq x_{t,n+1}(i,0)$.

**To prove (v)** consider (6). Let

$$z_i(k,t) = \begin{cases} x_{t,n}(i-k,0) \text{ if } k \leq i \\ -E \text{ if } k > i \end{cases} \quad \text{and} \quad Z_i(k) = \begin{cases} X_{0,n}(i-k,0) \text{ if } k \leq i \\ -E \text{ if } k > i \end{cases}$$

and condition on the demand that occurs in the next $t$ time units to get

$$
\begin{aligned}
x^E_{t+h,n+1}(i,0) - x^E_{t,n+1}(i,0) &= \int_0^t \lambda_2 e^{-\lambda_2 s} \sum_{k=0}^{i} e^{-\lambda_1 s} \frac{(\lambda_1 s)^k}{k!} \Big( x_{t+h-s,n}(i-k,0) - x_{t-s,n}(i-k,0) \Big)\, \mathrm{d}s \\
&- e^{-\lambda_2 t} \sum_{k_1=0}^{\infty} e^{-\lambda_1 t} \frac{(\lambda_1 t)^{k_1}}{k_1!} \Bigg[ Z_i(k_1) - \int_0^h \lambda_2 e^{-\lambda_2 s} \sum_{k_2=0}^{\infty} e^{-\lambda_1 s} \frac{(\lambda_1 s)^{k_2}}{k_2!} z_i(k_1+k_2, h-s)\mathrm{d}s \\
&- e^{-\lambda_2 h} \sum_{k_2=0}^{\infty} e^{-\lambda_1 h} \frac{(\lambda_1 h)^{k_2}}{k_2!} Z_i(k_1+k_2) \Bigg] \qquad (7)
\end{aligned}
$$

By the inductive hypothesis $x_{t,n}(i,0)$ is non-increasing in $t$, so the first integral in (7) is non-positive. (i), (ii) and (iii) imply that $z_i(k,t)$ and $Z_i(k)$ are non-increasing in $k$. Again $x_{t,n}(i,0)$ is non-increasing in $t$, so $z_i(k,t)$ is non-increasing in $t$. (iv) implies that $Z_i(k) \geq z_i(k,t)$. Combining these results we have $Z_i(k_1) \geq Z_i(k_1+k_2) \geq z_i(k_1+k_2, h-s)$ where $0 \leq s \leq h$. Thus the term in square brackets in (7) consists of $Z_i(k_1)$ less the weighted sum of functions which are all less than or equal to $Z_i(k_1)$. Since the weights are constructed from a probability density function, the integral of the weights is 1. Hence the term in square brackets is non-negative and $x^E_{t+h,n+1}(i,0) - x^E_{t,n+1}(i,0) \leq 0$. Therefore $x^E_{t,n+1}(i,0)$ is non-increasing in $t$ and hence $x^T_{t,n+1}(i,0)$ is non-increasing in $t$.

Recall that $w_{t,n+1}(0,0) = w^E_{t,n+1}(0,0)$ and for $i > 0$ $w_{t,n+1}(i,0) = w^T_{t,n+1}(i,0)$ if and only if $x^E_{t,n+1}(i-1,0) \geq T_{1,2} - E$. Since $x^E_{t,n+1}(i,0)$ is non-increasing in $t$, there exists $\tau_{i,n}$ such that $w_{t,n+1}(i,0) = w^T_{t,n+1}(i,0)$ if $0 < t \leq \tau_{i,n}$ and $w_{t,n+1}(i,0) = w^E_{t,n+1}(i,0)$ if $t > \tau_{i,n}$. From (ii) $x^E_{t,n+1}(i,0) \geq x^E_{t,n+1}(i-1,0)$, so $\tau_{i+1,n} \geq \tau_{i,n}$. Hence

$$
x_{t,n+1}(i,0) = \begin{cases} x^T_{t,n+1}(i,0) \text{ if } 0 < t \leq \tau_{i,n} \\ w^T_{t,n+1}(i+1,0) - w^E_{t,n+1}(i,0) = T_{1,2} - E \text{ if } \tau_{i,n} < t \leq \tau_{i+1,n} \\ x^E_{t,n+1}(i,0) \text{ if } \tau_{i+1,n} < t \leq 1 \end{cases}
$$

By definition $x^T_{t,n+1}(i,0) = x^E_{t,n+1}(i-1,0) \geq T_{1,2} - E$ if $t \leq \tau_{i,n}$ and $x^E_{n+1}(i,0) < T_{1,2} - E$ if $t > \tau_{i+1,n}$, so $x_{t,n+1}(i,0)$ is non-increasing in $t$ for $0 < t < 1$ and $0 \leq i < M_1$.

$\diamondsuit$